

THE LAPLACIAN ON THE UNIT SQUARE IN A SELF-SIMILAR MANNER

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ABSTRACT. In this paper, we show how to construct the standard Laplacian on the unit square in a self-similar manner. We rewrite the familiar mean value property of planar harmonic functions in terms of average values on small squares, from which we could know how the planar self-similar resistance form and the Laplacian look like. This approach combines the constructive limit-of-difference-quotients method of Kigami for p.c.f. self-similar sets and the method of averages introduced by Kusuoka and Zhou for the Sierpinski carpet.

1. INTRODUCTION

Let K denote a *self-similar set* which is non-empty compact and generated by a finite family of contraction similarity mappings $\{F_i\}_{i=1,\dots,N}$ on \mathbb{R}^n such that

$$K = \bigcup_{i=1}^N F_i K.$$

If $w = (w_1, \dots, w_m)$ is a finite word with each $w_j \in \{1, \dots, N\}$, we define the mapping

$$F_w = F_{w_1} \circ \dots \circ F_{w_m},$$

and call $F_w K$ a *m-cell* of K .

K is called *post-critically finite* (p.c.f.) if K is connected, and there exists a finite set $V_0 \subseteq K$ called the *boundary*, such that

$$F_w K \cap F_{w'} K \subseteq F_w V_0 \cap F_{w'} V_0, \quad (F_w V_0 \cap F_{w'} V_0) \cap V_0 = \emptyset, \\ \text{for } w \neq w' \text{ with } |w| = |w'|,$$

where $|w|$ is the length of w . Moreover, we require that each boundary point is the fixed point of one of the mappings in $\{F_i\}_{i=1,\dots,N}$. Without loss of generality we assume that $V_0 = \{q_1, \dots, q_{N_0}\}$ for $N_0 \leq N$ with

$$F_i q_i = q_i, \text{ for } i = 1, \dots, N_0.$$

A more general definition of p.c.f. self-similar sets can be found in [2,3] introduced by Kigami. The unit interval(I) and the Sierpinski gasket(SG) are two typical examples. However, the unit square (S , which we mainly discuss in this paper) and the Sierpinski carpet(SC) are non-p.c.f. self-similar sets.

An analytic construction of a Laplacian on SG was given by Kigami [1] as a *renormalized limit* of difference quotients, which he later extended to p.c.f. self-similar sets [2]. Please also refer to [6] to find a detailed introduction to this topic. At about the same time, Kusuoka and Zhou [4] developed a *method of average* for defining a Laplacian on SC , which uses average values of functions over cells

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rather than pointwise values in the definition. This method was later proved to be equivalent to Kigami's definition for *symmetric Laplacian* (with respect to the standard self-similar measure) by Strichartz [5] on SG , which brings the possibility to define Laplacians on more general self-similar sets.

Recall that SG is the invariant set of a family of 3 contraction mappings $F_i x = \frac{1}{3}(x + q_i)$, $i = 1, 2, 3$, where q_1, q_2, q_3 are the vertices of an equilateral triangle in \mathbb{R}^2 . Let μ be the standard *regular probability measure* with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ on SG . Define the average value for a function f on the cell $F_w SG$ as

$$B_w(f) = \frac{1}{\mu(F_w SG)} \int_{F_w SG} f d\mu.$$

It is easy to check that

$$B_w(f) = \frac{1}{3} \sum_{i=1}^3 B_{wi}(f),$$

where wi denotes (w_1, \dots, w_m, i) .

According to [5], any harmonic function h on SG satisfies the *mean value property*

$$B_w(h) = \frac{1}{3} \sum_{w' \sim_m w} B_{w'}(h)$$

for any finite word w with $F_w SG \cap V_0 = \emptyset$, where $|w'| = |w|$ and $w' \sim_m w$ means that $F_{w'} SG \cap F_w SG$ is non-empty.

The resistance form $\mathcal{E}(\cdot, \cdot)$ on SG is defined as

$$\mathcal{E}(f, g) = \lim_{m \rightarrow \infty} \mathcal{E}_m(f, g),$$

with

$$\mathcal{E}_m(f, g) = \frac{3}{2} r^{-m} \sum_{w' \sim_m w} (B_{w'}(f) - B_w(f))(B_{w'}(g) - B_w(g)),$$

where $r = \frac{3}{5}$ is the renormalization factor.

The symmetric Laplacian was proved to be the limit of a sequence of discrete Laplacians Δ_m , in the sense that

$$\Delta f(x) = \lim_{m \rightarrow \infty} \Delta_m f(x)$$

uniformly for every f in the domain of Δ , where Δ_m is defined as

$$\Delta_m f(x) = \frac{9}{2} r^{-m} 3^m \left(\frac{1}{3} \sum_{w' \sim_m w} B_{w'}(f) - B_w(f) \right), \text{ for } x \in F_w K, |w| = m,$$

for each integer $m \geq 0$.

In essential, the method of average studies first the analysis on *cell graphs* Γ_m whose vertices are the words w of length m , and edge relation, denoted by $w \sim_m w'$ is defined by the condition that $F_w SG \cap F_{w'} SG \neq \emptyset$, then passes the approximation to the limit as $m \rightarrow \infty$ to establish the analysis on SG .

In this paper, we will investigate that to what extent can we extend Kusuoka and Zhou's method to the planar unit square S , which could be viewed as a non-p.c.f. self-similar set, generated by a family of contraction mappings $\{F_i\}_{i=1}^4$ on \mathbb{R}^2 with

$$F_i(x) = \frac{1}{2}(x + q_i), \quad i = 1, 2, 3, 4,$$

and

$$q_1 = (0, 0), \quad q_2 = (1, 0), \quad q_3 = (1, 1), \quad q_4 = (0, 1).$$

In particular, the mean value property for planar harmonic functions on cell graphs (several choices) will not hold exactly. However, it will be close to holding exactly for large m . We will provide a sharp estimate for it. The analysis established in this paper provides a self-similar viewpoint of the familiar classical analysis on \mathbb{R}^2 . It will be nice to be able to transform the classical analytical results via this self-similar construction, which maybe will bring insights to the analysis on SC .

2. MEAN VALUE PROPERTY OF HARMONIC FUNCTIONS

From now on, we use μ to denote the *Lebesgue measure* on \mathbb{R}^2 . For $x = (x_1, x_2) \in \mathbb{R}^2$, $l \in \mathbb{R}^+$, denote

$$\mathcal{D}(x, l) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid x_1 - \frac{l}{2} \leq \xi_1 \leq x_1 + \frac{l}{2}, x_2 - \frac{l}{2} \leq \xi_2 \leq x_2 + \frac{l}{2}\} \quad (2.1)$$

a l -square centered at x . For a function f integrable on $\mathcal{D}(x, l)$, write

$$I(f, x, l) = \frac{1}{l^2} \int_{\mathcal{D}(x, l)} f d\mu. \quad (2.2)$$

Definition 2.1. Let $p = (p_1, p_2)$ be an integer pair with $0 \leq p_1 \leq p_2$. For two squares $\mathcal{D}(x, l)$ and $\mathcal{D}(x', l)$, write $\mathcal{D}(x, l) \sim_p \mathcal{D}(x', l)$ if

$$\begin{aligned} |x_1 - x'_1| &= p_1 l, \quad |x_2 - x'_2| = p_2 l, \quad \text{or} \\ |x_1 - x'_1| &= p_2 l, \quad |x_2 - x'_2| = p_1 l, \end{aligned} \quad (2.3)$$

call them p -neighbors.

Suppose f is integrable on every l -squares p -neighbor to $\mathcal{D}(x, l)$, denote

$$I_p(f, x, l) = \sum_{\substack{\mathcal{D}(x', l) \\ \sim_p \mathcal{D}(x, l)}} I(f, x', l). \quad (2.4)$$

For simplicity, denote a constant c_p as

$$c_p = \begin{cases} \frac{1}{8}, & \text{if } p_1 = p_2 = 0, \\ \frac{1}{2}, & \text{if } p_1 = 0, p_2 \neq 0, \text{ or } p_1 = p_2 \neq 0, \\ 1, & \text{if } 0 < p_1 < p_2. \end{cases} \quad (2.5)$$

It is easy to check that $8c_p$ is the number of p -neighbors of any fixed $\mathcal{D}(x, l)$.

Lemma 2.2. Let $x \in \mathbb{R}^2$, n be a non-negative integer. For functions f_n^x, g_n^x defined by

$$\begin{aligned} f_n^x(\xi) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j} (\xi_2 - x_2)^{2j}}{(n-2j)!(2j)!}, \\ g_n^x(\xi) &= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j-1} (\xi_2 - x_2)^{2j+1}}{(n-2j-1)!(2j+1)!}, \end{aligned} \quad (2.6)$$

and integer pair $p = (p_1, p_2)$ with $0 \leq p_1 \leq p_2$, we have

$$\begin{aligned} I_p(f_n^x, x, l) &= \frac{1}{(4k+2)!} T_p^{(k)} \left(\frac{l}{2}\right)^{4k}, \text{ if } 4 \mid n \text{ with } n = 4k, \\ I_p(f_n^x, x, l) &= 0, \text{ if } 4 \nmid n, \\ I_p(g_n^x, x, l) &= 0, \end{aligned} \tag{2.7}$$

where constants $T_p^{(k)}$ are

$$\begin{aligned} T_p^{(k)} &= 2c_p \left(\left((2p_1+1)^2 + (2p_2+1)^2 \right)^{2k+1} \sin \left((4k+2) \arctan \frac{2p_2+1}{2p_1+1} \right) \right. \\ &\quad - \left((2p_1-1)^2 + (2p_2+1)^2 \right)^{2k+1} \sin \left((4k+2) \arctan \frac{2p_2+1}{2p_1-1} \right) \\ &\quad + \left((2p_1-1)^2 + (2p_2-1)^2 \right)^{2k+1} \sin \left((4k+2) \arctan \frac{2p_2-1}{2p_1-1} \right) \\ &\quad \left. - \left((2p_1+1)^2 + (2p_2-1)^2 \right)^{2k+1} \sin \left((4k+2) \arctan \frac{2p_2-1}{2p_1+1} \right) \right), \end{aligned} \tag{2.8}$$

with the estimates that

$$|T_p^{(k)}| \leq 8c_p \|2p + (1, 1)\|^{4k+2}. \tag{2.9}$$

Proof. By direct calculation, we obtain

$$\begin{aligned} I_p(f_n^x, x, l) &= \\ &\frac{c_p}{l^2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{(n-2j+1)!(2j+1)!} \left(\left(\left((2p_1+1)\frac{l}{2} \right)^{n-2j+1} + \left(-(2p_1-1)\frac{l}{2} \right)^{n-2j+1} \right. \right. \\ &\quad - \left((2p_1-1)\frac{l}{2} \right)^{n-2j+1} - \left(-(2p_1+1)\frac{l}{2} \right)^{n-2j+1} \left(\left((2p_2+1)\frac{l}{2} \right)^{2j+1} \right. \\ &\quad + \left(-(2p_2-1)\frac{l}{2} \right)^{2j+1} - \left((2p_2-1)\frac{l}{2} \right)^{2j+1} - \left(-(2p_2+1)\frac{l}{2} \right)^{2j+1} \left. \right) \\ &\quad + \left(\left((2p_2+1)\frac{l}{2} \right)^{n-2j+1} + \left(-(2p_2-1)\frac{l}{2} \right)^{n-2j+1} - \left((2p_2-1)\frac{l}{2} \right)^{n-2j+1} \right. \\ &\quad - \left. \left. \left(-(2p_2+1)\frac{l}{2} \right)^{n-2j+1} \right) \left(\left((2p_1+1)\frac{l}{2} \right)^{2j+1} + \left(-(2p_1-1)\frac{l}{2} \right)^{2j+1} \right. \right. \\ &\quad \left. \left. - \left((2p_1-1)\frac{l}{2} \right)^{2j+1} - \left(-(2p_1+1)\frac{l}{2} \right)^{2j+1} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= c_p \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{4(n-2j+1)!(2j+1)!} \left(\frac{l}{2}\right)^n \left(\left(1 - (-1)^{n-2j+1}\right) \left((1 - (-1)^{2j+1})\right) \right. \\
&\quad \left((2p_1 + 1)^{n-2j+1} - (2p_1 - 1)^{n-2j+1} \right) \left((2p_2 + 1)^{2j+1} - (2p_2 - 1)^{2j+1} \right) \\
&\quad + \left(1 - (-1)^{n-2j+1}\right) \left((1 - (-1)^{2j+1})\right) \left((2p_2 + 1)^{n-2j+1} - (2p_2 - 1)^{n-2j+1} \right) \\
&\quad \left. \left((2p_1 + 1)^{2j+1} - (2p_1 - 1)^{2j+1} \right) \right).
\end{aligned}$$

Obviously the summation is zero unless n is even. If n is even, let $n = 2m$, then

$$I_p(f_n^x, x, l) = \frac{1}{(2m+2)!} \left(\frac{l}{2}\right)^{2m} c_p \sum_{j=0}^m (-1)^j C_{2m+2}^{2j+1} \gamma_p^{m-j,j},$$

with

$$\begin{aligned}
\gamma_p^{m-j,j} &= ((2p_1 + 1)^{2m-2j+1} - (2p_1 - 1)^{2m-2j+1}) ((2p_2 + 1)^{2j+1} - (2p_2 - 1)^{2j+1}) \\
&\quad + ((2p_2 + 1)^{2m-2j+1} - (2p_2 - 1)^{2m-2j+1}) ((2p_1 + 1)^{2j+1} - (2p_1 - 1)^{2j+1}).
\end{aligned}$$

Using the symmetry of $\sum_{j=0}^m (-1)^j C_{2m+2}^{2j+1} \gamma_p^{m-j,j}$, it is easy to check that the summation is zero unless m is even. If m is even, let $m = 2k$ ($n = 2m = 4k$), then we obtain that

$$I_p(f_n^x, x, l) = \frac{1}{(4k+2)!} T_p^{(k)} \left(\frac{l}{2}\right)^{4k},$$

and

$$\begin{aligned}
T_p^{(k)} &= c_p \sum_{j=0}^{2k} (-1)^j C_{4k+2}^{2j+1} \gamma_p^{2k-j,j} \\
&= c_p \sum_{j=0}^{2k} (-1)^j C_{4k+2}^{2j+1} \left(((2p_1 + 1)^{4k-2j+1} - (2p_1 - 1)^{4k-2j+1}) ((2p_2 + 1)^{2j+1} - (2p_2 - 1)^{2j+1}) \right. \\
&\quad \left. + ((2p_2 + 1)^{4k-2j+1} - (2p_2 - 1)^{4k-2j+1}) ((2p_1 + 1)^{2j+1} - (2p_1 - 1)^{2j+1}) \right) \\
&= c_p \left(Im(2p_1 + 1 + i(2p_2 + 1))^{4k+2} - Im(2p_1 - 1 + i(2p_2 + 1))^{4k+2} \right. \\
&\quad + Im(2p_1 - 1 + i(2p_2 - 1))^{4k+2} - Im(2p_1 + 1 + i(2p_2 - 1))^{4k+2} \\
&\quad + Im(2p_2 + 1 + i(2p_1 + 1))^{4k+2} - Im(2p_2 - 1 + i(2p_1 + 1))^{4k+2} \\
&\quad \left. + Im(2p_2 - 1 + i(2p_1 - 1))^{4k+2} - Im(2p_2 + 1 + i(2p_1 - 1))^{4k+2} \right).
\end{aligned}$$

Then an easy calculation yields (2.8) and (2.9).

Analogously, for g_n^x , a similar argument will gives $I_p(g_n^x, x, l) = 0$. \square

In fact, $\{f_n^x\}$ and $\{g_n^x\}$ are *polynomial harmonic functions* on \mathbb{R}^2 , which form a "basis" of harmonic functions near x .

Lemma 2.3. *Let p be an integer pair as before, Ω be an open set in \mathbb{R}^2 and $x \in \Omega$. Then there exists a positive constant $l_{x,p}$, such that for any harmonic function h on Ω and $l < l_{x,p}$,*

$$I_p(h, x, l) = 8c_p h(x) + \sum_{k=1}^{\infty} \frac{1}{(4k+2)!} \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) T_p^{(k)} \left(\frac{l}{2}\right)^{4k}, \quad (2.10)$$

with the same constants $T_p^{(k)}$ as in (2.8).

Proof. Since h is harmonic on Ω , it is real analytic in Ω . So we can expand h near x as

$$h(\xi) = \sum_{n=0}^{\infty} \sum_{i=0}^n \frac{\partial^n h}{\partial \xi_1^{n-i} \partial \xi_2^i}(x) \frac{(\xi_1 - x_1)^{n-i} (\xi_2 - x_2)^i}{(n-i)! i!}. \quad (2.11)$$

Noticing that

$$\Delta h(x) = \frac{\partial^2 h}{\partial \xi_1^2}(x) + \frac{\partial^2 h}{\partial \xi_2^2}(x) = 0,$$

we have

$$\frac{\partial^n h}{\partial \xi_1^{n-i} \partial \xi_2^i}(x) = -\frac{\partial^n h}{\partial \xi_1^{n-i-2} \partial \xi_2^{i+2}}(x), \text{ for } i = 0, 1, \dots, n-2.$$

By iterating we get

$$\begin{aligned} \frac{\partial^n h}{\partial \xi_1^{n-2j} \partial \xi_2^{2j}}(x) &= (-1)^j \frac{\partial^n h}{\partial \xi_1^n}(x), \text{ for } j = 0, 1, \dots, \left[\frac{n}{2}\right], \\ \frac{\partial^n h}{\partial \xi_1^{n-2j-1} \partial \xi_2^{2j+1}}(x) &= (-1)^j \frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2}(x), \text{ for } j = 0, 1, \dots, \left[\frac{n-1}{2}\right]. \end{aligned}$$

Then (2.11) gives

$$\begin{aligned} h(\xi) &= h(x) + \sum_{n=1}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{\partial^n h}{\partial \xi_1^{n-2j} \partial \xi_2^{2j}}(x) \frac{(\xi_1 - x_1)^{n-2j} (\xi_2 - x_2)^{2j}}{(n-2j)! (2j)!} \\ &\quad + \sum_{n=1}^{\infty} \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{\partial^n h}{\partial \xi_1^{n-2j-1} \partial \xi_2^{2j+1}}(x) \frac{(\xi_1 - x_1)^{n-2j-1} (\xi_2 - x_2)^{2j+1}}{(n-2j-1)! (2j+1)!} \\ &= h(x) + \sum_{n=1}^{\infty} \left(\frac{\partial^n h}{\partial \xi_1^n}(x) \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \frac{(\xi_1 - x_1)^{n-2j} (\xi_2 - x_2)^{2j}}{(n-2j)! (2j)!} \right) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2}(x) \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (-1)^j \frac{(\xi_1 - x_1)^{n-2j-1} (\xi_2 - x_2)^{2j+1}}{(n-2j-1)! (2j+1)!} \right). \\ &= h(x) + \sum_{n=1}^{\infty} \left(\frac{\partial^n h}{\partial \xi_1^n}(x) f_n^x(\xi) + \frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2}(x) g_n^x(\xi) \right), \end{aligned} \quad (2.12)$$

where f_n^x and g_n^x are same as those in Lemma 2.2.

Choose

$$l_{x,p} = \sup\{l > 0 \mid \text{All } p\text{-neighbors of } \mathcal{D}(x, l) \text{ are contained in the convergence domain of (2.11)}\}.$$

Then for any $l < l_{x,p}$, we have

$$I_p(h, x, l) = 8c_p h(x) + \sum_{n=1}^{\infty} \left(\frac{\partial^n h}{\partial \xi_1^n}(x) I_p(f_n^x, x, l) + \frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2}(x) I_p(g_n^x, x, l) \right). \quad (2.13)$$

By using Lemma 2.2, this gives (2.10). \square

The following is the *mean value property* of planar harmonic functions in terms of average values on l -squares.

Theorem 2.4. *Suppose \mathcal{P} is a finite set consisting of integer pairs $p = (p_1, p_2)$ with $0 \leq p_1 \leq p_2$ and $p_2 \neq 0$, and $\{A_p\}_{p \in \mathcal{P}}$ is a collection of real numbers satisfies*

$$8 \sum_{p \in \mathcal{P}} c_p A_p = 1. \quad (2.14)$$

Let Ω be an open set in \mathbb{R}^2 and $x \in \Omega$. Then for any harmonic function h on Ω and $l < l_{x,\mathcal{P}}$, we have

$$|I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l)| \leq \sum_{k=1}^{\infty} \frac{1}{(4k+2)!} \left| \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \right| \left(2^{2k+1} + \|\mathcal{P}\|^{4k+2} \right) \left(\frac{l}{2} \right)^{4k}, \quad (2.15)$$

where

$$\|\mathcal{P}\| = \max_{p \in \mathcal{P}} \|2p + (1, 1)\|, \quad (2.16)$$

and $l_{x,\mathcal{P}} = \min_{p \in \mathcal{P}} l_{x,p}$ with $l_{x,p}$ being the same as in Lemma 2.3.

Proof. Noticing that

$$I(h, x, l) = I_{\theta}(h, x, l), \text{ with } \theta = (0, 0),$$

by using (2.10), we obtain that

$$\begin{aligned} & |I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l)| \\ &= \left| \left(1 - 8 \sum_{p \in \mathcal{P}} c_p A_p \right) h(x) + \sum_{k=1}^{\infty} \frac{1}{(4k+2)!} \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \left(T_{\theta}^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right) \left(\frac{l}{2} \right)^{4k} \right| \\ &= \sum_{k=1}^{\infty} \frac{1}{(4k+2)!} \left| \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \right| \left| T_{\theta}^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right| \left(\frac{l}{2} \right)^{4k}. \end{aligned}$$

By calculation $T_{\theta}^{(k)} = (-1)^k 2^{2k+1}$ and using the estimate (2.9) we then have

$$\begin{aligned} & \left| T_{\theta}^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right| \\ & \leq 2^{2k+1} + \sum_{p \in \mathcal{P}} 8c_p A_p \|2p + (1, 1)\|^{4k+2} \\ & \leq 2^{2k+1} + \|\mathcal{P}\|^{4k+2}, \end{aligned}$$

which gives (2.15). \square

Remark. In fact, given \mathcal{P} , by choosing A_p properly, we can get “higher” rate of convergence for (2.15). Define

$$N = \inf \{ k \in \mathbb{N} \mid \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \neq T_{\theta}^{(k)} \}, \quad (2.17)$$

called the *mean value level* of \mathcal{P} with coefficients $\{A_p\}_{p \in \mathcal{P}}$. Then (2.15) becomes

$$|I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l)| \leq \sum_{k=N}^{\infty} \frac{1}{(4k+2)!} \left| \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \right| \left(2^{2k+1} + \|\mathcal{P}\|^{4k+2} \right) \left(\frac{l}{2} \right)^{4k}. \quad (2.18)$$

We will discuss more on the mean value level in Section 4.

3. THE RESISTANCE FORM AND THE LAPLACIAN ON THE UNIT SQUARE S

In this section, we will show the expressions of the *resistance form* and the (*symmetric*) *Laplacian* on the unit square S in terms of average values on cells.

Lemma 3.1. *Let \mathcal{P} and $\{A_p\}_{p \in \mathcal{P}}$ be defined as in Theorem 2.4, Ω be an open set in \mathbb{R}^2 . Then for any $f \in C^1(\Omega)$ and $x \in \Omega$, we have*

$$|\nabla f(x)|^2 = \frac{1}{2} \mathcal{M}_{\mathcal{P}} \lim_{l \rightarrow 0} \frac{1}{l^2} \sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{\mathcal{D}(x', l) \\ \sim_p \mathcal{D}(x, l)}} (I(f, x', l) - I(f, x, l))^2 \right), \quad (3.1)$$

where

$$\mathcal{M}_{\mathcal{P}} = \frac{1}{2} \left(\sum_{p \in \mathcal{P}} \|p\|^2 c_p A_p \right)^{-1}. \quad (3.2)$$

Proof. Let $l < l_{x, \mathcal{P}}$ where $l_{x, \mathcal{P}}$ is the same as defined in Theorem 2.4. Then for each $p \in \mathcal{P}$, there are $8c_p$ p -neighbors of the l -square $\mathcal{D}(x, l)$. It is easy to check that $\mathcal{D}(x', l)$ with $x' = (x_1 + p_1 l, x_2 + p_2 l)$ is one of them. Using the mean value theorem for integral we have

$$\begin{aligned} I(f, x', l) - I(f, x, l) &= \frac{1}{l^2} \int_{\mathcal{D}(x', l)} f(\xi_1, \xi_2) d\mu(\xi) - \frac{1}{l^2} \int_{\mathcal{D}(x, l)} f(\xi_1, \xi_2) d\mu(\xi) \\ &= \frac{1}{l^2} \int_{\mathcal{D}(x, l)} \left(f(\xi_1 + p_1 l, \xi_2 + p_2 l) - f(\xi_1, \xi_2) \right) d\mu(\xi) \\ &= f(\eta_1 + p_1 l, \eta_2 + p_2 l) - f(\eta_1, \eta_2) \end{aligned}$$

for some $(\eta_1, \eta_2) \in \mathcal{D}(x, l)$. Hence

$$\lim_{l \rightarrow 0} \frac{1}{l} (I(f, x', l) - I(f, x, l)) = p_1 \frac{\partial f}{\partial \xi_1}(x) + p_2 \frac{\partial f}{\partial \xi_2}(x).$$

Dealing with other p -neighbors similarly, and summing over all the $8c_p$ terms, we get

$$\begin{aligned} &\lim_{l \rightarrow 0} \sum_{\substack{\mathcal{D}(x', l) \\ \sim_p \mathcal{D}(x, l)}} \frac{1}{l^2} (I(f, x', l) - I(f, x, l))^2 \\ &= c_p \left(\left(p_1 \frac{\partial f}{\partial \xi_1}(x) + p_2 \frac{\partial f}{\partial \xi_2}(x) \right)^2 + \left(p_2 \frac{\partial f}{\partial \xi_1}(x) + p_1 \frac{\partial f}{\partial \xi_2}(x) \right)^2 + \left(-p_1 \frac{\partial f}{\partial \xi_1}(x) + p_2 \frac{\partial f}{\partial \xi_2}(x) \right)^2 \right. \\ &\quad + \left(-p_2 \frac{\partial f}{\partial \xi_1}(x) + p_1 \frac{\partial f}{\partial \xi_2}(x) \right)^2 + \left(-p_1 \frac{\partial f}{\partial \xi_1}(x) - p_2 \frac{\partial f}{\partial \xi_2}(x) \right)^2 + \left(-p_2 \frac{\partial f}{\partial \xi_1}(x) - p_1 \frac{\partial f}{\partial \xi_2}(x) \right)^2 \\ &\quad \left. + \left(p_1 \frac{\partial f}{\partial \xi_1}(x) - p_2 \frac{\partial f}{\partial \xi_2}(x) \right)^2 + \left(p_2 \frac{\partial f}{\partial \xi_1}(x) - p_1 \frac{\partial f}{\partial \xi_2}(x) \right)^2 \right) \\ &= 4\|p\|^2 c_p |\nabla f(x)|^2. \end{aligned}$$

Summing over all $p \in \mathcal{P}$ with the coefficient A_p , we obtain (3.1). \square

Now we turn to the resistance form on the unit square S . Notice that the Lebesgue measure μ restricted to S becomes a regular probability measure with equal weights. Analogous to the SG case, for a finite word $w = (w_1, \dots, w_m)$ with each $w_j \in \{1, 2, 3, 4\}$, define the average value for a function f on $F_w S$ as

$$B_w(f) = \frac{1}{\mu(F_w S)} \int_{F_w S} f d\mu.$$

Theorem 3.2. *Let \mathcal{P} and $\{A_p\}_{p \in \mathcal{P}}$ be defined as before. For any $f, g \in C^1(S)$, $m \geq 0$, define*

$$\mathcal{E}_m(f, g) = \mathcal{M}_{\mathcal{P}} \sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{|w|=|w'|=m, \\ F_{w'} S \sim_p F_w S}} (B_{w'}(f) - B_w(f))(B_{w'}(g) - B_w(g)) \right), \quad (3.3)$$

where $\mathcal{M}_{\mathcal{P}}$ is the same as (3.2). Then we have

$$\mathcal{E}(f, g) := \lim_{m \rightarrow \infty} \mathcal{E}_m(f, g) = \int_S \nabla f \cdot \nabla g d\mu. \quad (3.4)$$

Proof. For a m -cell $F_w S$ in the unit square S , let l_m denote the side length and x_w the center of $F_w S$. Then

$$l_m = \frac{1}{2^m}, \quad \mu(F_w S) = \frac{1}{4^m}, \quad \text{and } x_w = F_w \left(\frac{1}{4} \sum_i q_i \right),$$

hence

$$B_w(f) = I(f, x_w, l_m), \quad F_w S = \mathcal{D}(x_w, l_m).$$

Noticing that for $p \in \mathcal{P}$, the p -neighbors of $F_w S$ may not be within S , we define

$$S_m = \bigcup \{F_w S : |w| = m \text{ and all } p\text{-neighbors of } F_w S \text{ are contained in } S\},$$

and $S_m^c = S \setminus S_m$. Then we have

$$\begin{aligned} \mathcal{E}_m(f, f) &= \mathcal{M}_{\mathcal{P}} \sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{|w|=|w'|=m, \\ F_{w'} S \sim_p F_w S}} (B_{w'}(f) - B_w(f))^2 \right) \\ &= \frac{1}{2} \mathcal{M}_{\mathcal{P}} \sum_{|w|=m} \sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{F_{w'} S \\ \sim_p F_w S}} (B_{w'}(f) - B_w(f))^2 \right) \\ &= \frac{1}{2} \mathcal{M}_{\mathcal{P}} \sum_{\substack{|w|=m \\ F_w S \subseteq S_m}} \left(\frac{1}{l_m^2} \sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{\mathcal{D}(x_{w'}, l_m) \\ \sim_p \mathcal{D}(x_w, l_m)}} (I(f, x_{w'}, l_m) - I(f, x_w, l_m))^2 \right) \mu(F_w S) \right) \\ &\quad + \frac{1}{2} \mathcal{M}_{\mathcal{P}} \sum_{\substack{|w|=m \\ F_w S \subseteq S_m^c}} \left(\frac{1}{l_m^2} \sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{\mathcal{D}(x_{w'}, l_m) \\ \sim_p \mathcal{D}(x_w, l_m)}} (I(f, x_{w'}, l_m) - I(f, x_w, l_m))^2 \right) \mu(F_w S) \right). \end{aligned}$$

It is obvious that $\lim_{m \rightarrow \infty} \mu(S_m^c) = 0$. So by Lemma 3.1 we know that when m goes to ∞ , the first term converges to $\int_S |\nabla f|^2 d\mu$ and the second term converges to 0. Hence

$$\mathcal{E}(f, f) = \lim_{m \rightarrow \infty} \mathcal{E}_m(f, f) = \int_S |\nabla f|^2 d\mu.$$

Then by using the polarization identity

$$\mathcal{E}_m(f, g) = \frac{1}{4} \left(\mathcal{E}_m(f + g, f + g) - \mathcal{E}_m(f - g, f - g) \right), \quad (3.5)$$

we obtain

$$\begin{aligned} \mathcal{E}(f, g) &= \lim_{m \rightarrow \infty} \mathcal{E}_m(f, g) \\ &= \frac{1}{4} \left(\int_K |\nabla(f + g)|^2 d\mu - \int_K |\nabla(f - g)|^2 d\mu \right) \\ &= \int_K \nabla f \cdot \nabla g d\mu. \end{aligned}$$

□

We should remark that (3.3) and (3.4) imply that the renormalization factor of the resistance form equals to 1 in this case.

Next we come to the Laplacian on S .

Lemma 3.3. *Let \mathcal{P} and $\{A_p\}_{p \in \mathcal{P}}$ be defined as before. Ω be an open set in \mathbb{R}^2 . Then for any $f \in C^2(\Omega)$ and $x \in \Omega$, we have*

$$\Delta f(x) = \mathcal{M}_{\mathcal{P}} \lim_{l \rightarrow 0} \frac{1}{l^2} \left(\sum_{p \in \mathcal{P}} A_p I_p(f, x, l) - I(f, x, l) \right), \quad (3.6)$$

where $\mathcal{M}_{\mathcal{P}}$ is the same as (3.2).

Proof. Since $8 \sum_{p \in \mathcal{P}} c_p A_p = 1$, we have

$$\begin{aligned} \sum_{p \in \mathcal{P}} A_p I_p(f, x, l) - I(f, x, l) &= \sum_{p \in \mathcal{P}} A_p \left(I_p(f, x, l) - 8c_p I(f, x, l) \right) \\ &= \sum_{p \in \mathcal{P}} A_p \left(\sum_{\substack{\mathcal{D}(x', l) \\ \sim_p \mathcal{D}(x, l)}} (I(f, x', l) - I(f, x, l)) \right). \end{aligned}$$

Let $p \in \mathcal{P}$. As in the proof of Lemma 3.1, there are $8c_p$ p -neighbors of the l -square $\mathcal{D}(x, l)$, for example, $\mathcal{D}(x', l)$ and $\mathcal{D}(x'', l)$ with $x' = (x_1 + p_1 l, x_2 + p_2 l)$ and $x'' = (x_1 - p_1 l, x_2 - p_2 l)$ are two of them. By the mean value theorem for integral, we have

$$\begin{aligned} &I(f, x', l) - I(f, x, l) + I(f, x'', l) - I(f, x, l) \\ &= f(\eta_1 + p_1 l, \eta_2 + p_2 l) - f(\eta_1, \eta_2) + f(\eta_1 - p_1 l, \eta_2 - p_2 l) - f(\eta_1, \eta_2), \text{ for some } (\eta_1, \eta_2) \in \mathcal{D}(x, l). \end{aligned}$$

Thus

$$\lim_{l \rightarrow 0} \frac{1}{l^2} \left(I(f, x', l) - I(f, x, l) + I(f, x'', l) - I(f, x, l) \right) = p_1^2 \frac{\partial^2 f}{\partial \xi_1^2}(x) + 2p_1 p_2 \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2}(x) + p_2^2 \frac{\partial^2 f}{\partial \xi_2^2}(x).$$

Dealing with other p -neighbors similarly, and summing over all of them, we obtain

$$\begin{aligned}
& \lim_{l \rightarrow 0} \frac{1}{l^2} \sum_{\substack{\mathcal{D}(x', l) \\ \sim_p \mathcal{D}(x, l)}} (I(f, x', l) - I(f, x, l)) \\
&= c_p \left(p_1^2 \frac{\partial^2 f}{\partial \xi_1^2}(x) + 2p_1 p_2 \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2}(x) + p_2^2 \frac{\partial^2 f}{\partial \xi_2^2}(x) + p_2^2 \frac{\partial^2 f}{\partial \xi_1^2}(x) + 2p_1 p_2 \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2}(x) + p_1^2 \frac{\partial^2 f}{\partial \xi_2^2}(x) \right. \\
&\quad + (-p_1)^2 \frac{\partial^2 f}{\partial \xi_1^2}(x) + 2(-p_1)p_2 \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2}(x) + p_2^2 \frac{\partial^2 f}{\partial \xi_2^2}(x) + (-p_2)^2 \frac{\partial^2 f}{\partial \xi_1^2}(x) + 2p_1(-p_2) \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2}(x) \\
&\quad \left. + p_1^2 \frac{\partial^2 f}{\partial \xi_2^2}(x) \right) \\
&= 2\|p\|^2 c_p \Delta f(x).
\end{aligned}$$

Summing over all $p \in \mathcal{P}$, we obtain (3.6). \square

Then we could apply Lemma 3.3 to the unit square S to reconstruct the Laplacian in a self-similar manner.

Theorem 3.4. *Let \mathcal{P} and $\{A_p\}_{p \in \mathcal{P}}$ be defined as before. For any $f \in C^2(S)$, $x \in S \setminus \partial S$, $m \geq 0$, define*

$$\Delta_m f(x) = \mathcal{M}_{\mathcal{P}} 4^m \left(\sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{F_{w'} S \\ \sim_p F_w S}} B_{w'}(f) \right) - B_w(f) \right) \text{ for } x \in F_w S \text{ with } |w| = m, \quad (3.7)$$

where $\mathcal{M}_{\mathcal{P}}$ is the same as (3.2). Then

$$\Delta f(x) = \lim_{m \rightarrow \infty} \Delta_m f(x) \quad (3.8)$$

uniformly.

Proof. Similarly, we define S_m and S_m^c as we did in the proof of Theorem 3.2. Since $x \in S \setminus \partial S$ we know that there exists an integer m_0 such that when $m \geq m_0$ (that is, $l_m \leq 1/2^{m_0}$), $x \in F_w S \subseteq S_m$ for some w of length m . Obviously, x_w (the center of $F_w S$) will go to x as m goes to ∞ . Then from (3.6) we get

$$\begin{aligned}
\lim_{m \rightarrow \infty} \Delta_m f(x) &= \mathcal{M}_{\mathcal{P}} \lim_{m \rightarrow \infty} 4^m \left(\sum_{p \in \mathcal{P}} \left(A_p \sum_{\substack{F_{w'} S \\ \sim_p F_w S}} B_{w'}(f) \right) - B_w(f) \right) \\
&= \mathcal{M}_{\mathcal{P}} \lim_{l_m \rightarrow 0} \frac{1}{l_m^2} \left(\sum_{p \in \mathcal{P}} A_p I_p(f, x_w, l_m) - I(f, x_w, l_m) \right) \\
&= \Delta f(x).
\end{aligned}$$

The uniform convergence comes from the fact that S is compact. \square

4. THE LEVEL OF MEAN VALUE PROPERTY

In Section 2 we have seen that the rate of convergence of (2.15) is decided by the mean value level of $(\mathcal{P}, \{A_p\}_{p \in \mathcal{P}})$. For given \mathcal{P} and positive integer N , from (2.17), we may find $\{A_p\}_{p \in \mathcal{P}}$ by solving equations

$$\begin{aligned}
8 \sum_{p \in \mathcal{P}} c_p A_p &= 1, \\
\sum_{p \in \mathcal{P}} A_p T_p^{(k)} &= T_\theta^{(k)}, \text{ for } k = 0, 1, \dots, N-1,
\end{aligned} \tag{4.1}$$

such that N is the mean value level of $(\mathcal{P}, \{A_p\}_{p \in \mathcal{P}})$.

Here are some solutions:

For $\mathcal{P} = \{(0, 1)\}$, $N = 1$, we have a unique solution

$$A_{(0,1)} = \frac{1}{4}, \mathcal{M}_{\mathcal{P}} = 4;$$

For $\mathcal{P} = \{(0, 1), (1, 1)\}$, $N = 1$, we have infinite solutions satisfying

$$A_{(0,1)} + A_{(1,1)} = \frac{1}{4};$$

For $\mathcal{P} = \{(0, 1), (1, 1)\}$, $N = 2$, we have a unique solution

$$A_{(0,1)} = \frac{1}{5}, A_{(1,1)} = \frac{1}{20}, \mathcal{M}_{\mathcal{P}} = \frac{10}{3};$$

For $\mathcal{P} = \{(0, 1), (1, 1), (0, 2)\}$, $N = 3$, we have a unique solution

$$A_{(0,1)} = \frac{16}{75}, A_{(1,1)} = \frac{1}{25}, A_{(0,2)} = -\frac{1}{300}, \mathcal{M}_{\mathcal{P}} = \frac{25}{7};$$

For $\mathcal{P} = \{(0, 1), (1, 1), (0, 2), (1, 2)\}$, $N = 4$, we have a unique solution

$$A_{(0,1)} = \frac{38}{183}, A_{(1,1)} = \frac{103}{2379}, A_{(0,2)} = -\frac{17}{9516}, A_{(1,2)} = \frac{1}{2379}, \mathcal{M}_{\mathcal{P}} = \frac{793}{231}.$$

On the other hand, for a given \mathcal{P} with coefficients $\{A_p\}_{p \in \mathcal{P}}$, it is natural to ask which kind of harmonic functions does satisfy the mean value property exactly.

Theorem 4.1 *Let $(\mathcal{P}, \{A_p\}_{p \in \mathcal{P}})$ be defined as before with the mean value level N , Ω be a connected open set in \mathbb{R}^2 , and h be a harmonic function on Ω . Suppose there is a connected open subset U of Ω such that for any $x \in U$, the identity*

$$I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l) = 0$$

holds for sufficiently small l . Then h is a polynomial harmonic function on Ω with degree no more than $4N$.

Proof. By applying (2.10) and (2.17), we know that for any $x \in U$ and sufficiently small l ,

$$I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l) = \sum_{k=N}^{\infty} \frac{1}{(4k+2)!} \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \left(T_\theta^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right) \left(\frac{l}{2} \right)^{4k} = 0.$$

From the arbitrariness of l , we know that for any $k \geq N$,

$$\frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \left(T_\theta^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right) = 0.$$

Then by the definition of N ,

$$T_\theta^{(N)} - \sum_{p \in \mathcal{P}} A_p T_p^{(N)} \neq 0,$$

thus

$$\frac{\partial^{4N} h}{\partial \xi_1^{4N}}(x) = 0, \forall x \in U.$$

Hence from (2.12), we can expand h near x as a polynomial harmonic function of degree no more than $4N$ as follows

$$\begin{aligned} h(\xi) = & h(x) + \sum_{n=1}^{4N-1} \left(\frac{\partial^n h}{\partial \xi_1^n}(x) \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j}}{(n-2j)!} \frac{(\xi_2 - x_2)^{2j}}{(2j)!} \right) \\ & + \sum_{n=1}^{4N} \left(\frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2}(x) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j-1}}{(n-2j-1)!} \frac{(\xi_2 - x_2)^{2j+1}}{(2j+1)!} \right). \end{aligned}$$

Since h is harmonic on Ω , h is real analytic in Ω . Then from the principle of analytic continuation, we know that the above identity is valid for any $x \in \Omega$. \square

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